An Operational Definition of $\underset{\sim}{F}=m\underset{\sim}{a}$

An operational definition of mass m and acceleration $\underset{\sim}{a}$ are given without invoking the notion of force $\underset{\sim}{F}$ and thus Newton's force law, $\underset{\sim}{F}=m\underset{\sim}{a}$ can be operationally defined non-recursively.

Start with an orthogonal x,y,z, coordinate inertial frame of reference, where an inertial frame is an non-accelerating frame as measured by onboard accelerometers. If the accelerometers register zero, the accelerometers are in an inertial frame. A freely falling elevator in a gravitational field would appear to be in an inertial frame as onboard accelerometers would be weightless and register zero acceleration. However, the divergence of the gravitational field will exert a small compressive force at right angles to the direction of motion on the accelerometers. The accelerometers will register a nonzero acceleration and the freely falling elevator will not be in an inertial frame.

The operational definition of vector force $\underset{\sim}{F}=m\underset{\sim}{a}$ is operationally defined as follows.

1. Gravitational mass m is operationally defined in terms of a given standard mass call it 1kg. By balancing unknown weights against 1kg using a balance and by cutting off enough material of the unknown until balance is reached, multiple copies of 1kg may be made. $\frac{1}{2}$ of 1kg may be made by cutting 1kg in half and balancing both halves to check that one has accurately cut 1kg in half. Repeat the process to make multiple copies of $\frac{1}{2}$ of 1kg. By multiple repetitions of this process, multiple copies may be made of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... of 1kg. By trial and error, balance unknown mass m against the proper number of fractional equivalents of 1kg to determine the mass of m. Mass determined in this way is called the gravitational mass.
2. Make a Hook's law spring and check that it is a Hook's law spring by hanging mass m_1 from it, resulting in a change of length ΔL. Then hang $2m_1$ from it, and if it is a Hook's law spring, $2\Delta L$ will result. Repeat with m_2 and with enough different m_i's until one is convinced that one has made a hook's law spring. Notice that the notion of force F or the formulae $F=-K\Delta L$ or $F=ma$ have not been used in the construction of the Hook's law spring.
3. Hang the mass m from the spring and note ΔL_m.
4. Vector acceleration is measured from an inertial frame and is mathematically defined as $\underset{\sim}{a}=\lim_{\Delta t\to 0}\frac{\Delta^2\underset{\sim}{x}}{\Delta t^2}$ and operationally defined as $\underset{\sim}{a}=\lim_{\Delta t\to \varepsilon}\frac{\Delta^2\underset{\sim}{x}}{\Delta t^2}$ where ε is $\varepsilon>0$ but small enough so that changing ε to $\varepsilon\pm0.1\varepsilon$ results in a change in the value of each component of $\underset{\sim}{a}$ that is small enough to be ignored.
5. Release mass m in 3 above from the spring and measure the acceleration g. g experimentally turns out to be a constant near the surface of the earth and independent of mass m.
6. Construct a frictionless table so that pushing mass m up to speed v and releasing it, results in speed $v-\Delta v$ where Δv is small enough to be ignored as m moves across the length of the table.

7. Placing mass m on the frictionless table, pull the mass with the Hook's law spring parallel to the table surface and in a constant direction with the spring stretched by ΔL_m as in 3 above. Mass m in this experiment is called the inertial mass.
8. Measure the acceleration of m in the experimental setup described in 7. Call it h.
9. Define $\underset{\sim}{F}=m\underset{\sim}{a}$ and in particular if $\underset{\sim}{a}$ is in a constant direction, then F=ma. Call F the linear force.
10. Define the gravitational linear force $F_g=-K\Delta L=m_g g$ and the inertial linear force $F_{in}=-K\Delta L=m_{in}h$ where m_g is the gravitational mass defined above and m_{in} is the inertial mass operationally defined by $m_{in}=-\frac{K\Delta L}{h}$. F_g is an action at a distance force and F_{in} is a contact force. If $h\neq g$ then $m_g\neq m_{in}$ and gravitational mass does not equal inertial mass. However experimentally, h=g so $m_g=m_{in}$ and therefore gravitational mass equals inertial mass and the gravitational linear force equals the inertial linear force $F_g=F_{in}$.
An operational definition has been given of Newton's force law, $\underset{\sim}{F}=m\underset{\sim}{a}$.