

Chapter 2. Coordinate Transforms

1. Moving Point Versus Stationary Point- Accelerating Point as Seen from S

Let \underline{x} represent a rectilinear coordinate point stationary in inertial frame S where all measurements of the position of \underline{x} at time t are taken by observers stationary in S using clocks stationary in S. It is quite common to allow \underline{x} to be a function of t,

$\underline{x}=\underline{x}(t)$ with $\underline{v}=(v_x,v_y,v_z)=\frac{d\underline{x}}{dt}$ and $\underline{a}=(a_x,a_y,a_z)=\frac{d^2\underline{x}}{dt^2}$ e.g. The traditional derivation of

the oceanic tidal amplitude and the mathematical theory of fluid turbulence make use of the $(\underline{U}\cdot\nabla)\underline{U}$ term occurring in the Navier-Stokes Equations with

$\underline{U}=\underline{U}[x(t),y(t),z(t),t]$ i.e. the derivation explicitly assumes that x,y,z are functions of t,

$\underline{x}=\underline{x}(t)$, $\underline{U}=\frac{d\underline{x}}{dt}$ and $\underline{a}=\frac{d^2\underline{x}}{dt^2}$.

With $\underline{U}=U_x\hat{x}+U_y\hat{y}+U_z\hat{z}$, take the derivative of \underline{U} w.r.t. t:

$$2.1a. \frac{d\underline{U}}{dt}=\frac{\partial\underline{U}}{\partial t}+\frac{\partial\underline{U}}{\partial x}\frac{dx}{dt}+\frac{\partial\underline{U}}{\partial y}\frac{dy}{dt}+\frac{\partial\underline{U}}{\partial z}\frac{dz}{dt}=\frac{\partial\underline{U}}{\partial t}+(\underline{U}\cdot\nabla)\underline{U}.$$

All x, y, z in these expressions are functions of t. Although this expression is mathematically correct, it is not physically correct. Newton's Laws as exemplified by

$\rho(\underline{x},t)\cdot\frac{d\underline{U}}{dt}=-\nabla P$, or $\underline{F}=\frac{d}{dt}(m\underline{U})$ are valid only if position, velocity and acceleration are measured from a frame S which is determined to be inertial as determined by on board accelerometers at rest in S. To be physically correct, the \underline{x} occurring in $\underline{U}=\frac{d\underline{x}}{dt}$

must be stationary in inertial frame S and therefore independent of time t as measured by observers at rest w.r.t. clocks at rest in S. The operational definition of \underline{U} is thus inconsistent with the mathematical definition of \underline{U} i.e. $\underline{U}=\frac{d\underline{x}(\underline{x}_0,t)}{dt}$. The

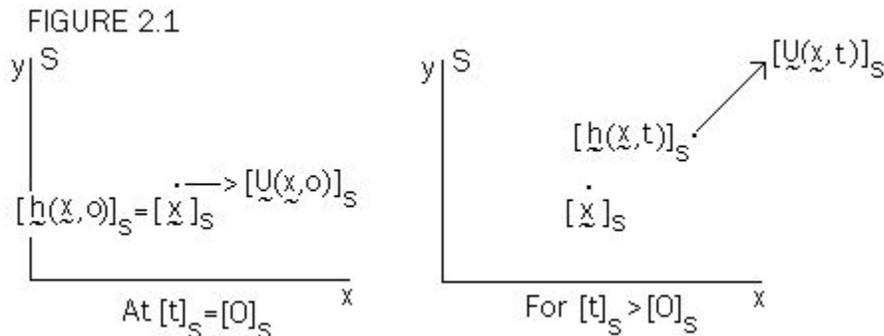
physically correct expression for \underline{U} is $\underline{U}=\frac{d\underline{h}}{dt}$ where \underline{h} is defined below.

As we are interested in measuring a moving object from both inertial frame S and S', a more general notation than $\underline{U}=\frac{d\underline{h}}{dt}$ will be used.

Consider a point moving arbitrarily within inertial frame S. All determinations of position and time of the moving point are made by observers $O_{\underline{x}}$ momentarily coincident with the moving point at time t. Observer $O_{\underline{x}}$ is at rest in \underline{x} in S using clock $K_{\underline{x}}$ at rest at \underline{x} in S. Clocks in S are synchronized with clocks in S in the usual way. Let $[\underline{x}]_S$, fixed in S, represent the position of a moving point at time $[t]_S=[0]_S$ and let the function $[\underline{h}]_S=([h_1]_S, [h_2]_S, [h_3]_S)$ represent the position of the same moving point at arbitrary time $[t]_S$. See figure 2.1. It is understood when using this notation

that all observations of position and time are made by observers $O_{\tilde{x}}$ at rest in S. Let $[\tilde{x}(\tilde{x}, t)]_S$ be a function such that:

$$2.1b. \quad [h(\tilde{x}, t)]_S = [\tilde{x}]_S + [\tilde{x}(\tilde{x}, t)]_S \quad \text{with} \quad [\tilde{x}(\tilde{x}, 0)]_S = [Q]_S$$



$[h]_S = [h(\tilde{x}, t)]_S$ will replace the symbol $\tilde{x}(t)$. A distinction must be made between $[\Delta\tilde{x}]_S = [\tilde{x}_2]_S - [\tilde{x}_1]_S$ with $[\tilde{x}_2]_S$ and $[\tilde{x}_1]_S$ fixed in S and $[\Delta\tilde{x}]_S = [h(\tilde{x}, t + \Delta t)]_S - [h(\tilde{x}, t)]_S$. In the rest of this text, $[\Delta\tilde{x}]_S$ will always mean $[\Delta\tilde{x}]_S = [\tilde{x}_2]_S - [\tilde{x}_1]_S$ with $[\tilde{x}_2]_S$ and $[\tilde{x}_1]_S$ fixed in S. If a point is moving with respect to S, then the change in position of that point as a function of time is given by $[h(\tilde{x}, t + \Delta t)]_S - [h(\tilde{x}, t)]_S$ with $h(\tilde{x}, 0) = 0$. With this

notation: $\left[\frac{d[\tilde{x}]_S}{dt}\right]_S \equiv \left[\frac{d[\tilde{x}]_S}{[dt]_S}\right]_S = \lim_{[\Delta t]_S \rightarrow 0} \frac{[\Delta[\tilde{x}]_S]_S}{[\Delta t]_S} = \lim_{[\Delta t]_S \rightarrow 0} \frac{[[\tilde{x}_2]_S - [\tilde{x}_1]_S]_S}{[\Delta t]_S} \equiv [Q]_S$. The

special case of coordinate frame transformations is discussed in chapter 2, sections 5 and 6.

The velocity $[U]_S$ of the arbitrarily moving point with respect to S is: $[U]_S = \left[\frac{dh}{dt}\right]_S$

and the acceleration of the moving point is: $[a]_S = \left[\frac{dU}{dt}\right]_S$ with:

$$2.2 \quad \left[\frac{\partial h}{\partial t}\right]_S = \left[\frac{\partial \tilde{x}(\tilde{x}, t)}{\partial t}\right]_S \quad \text{and} \quad \left[\frac{\partial^2 h}{\partial t^2}\right]_S = \left[\frac{\partial^2 \tilde{x}(\tilde{x}, t)}{\partial t^2}\right]_S$$

and in all cases, $\left[\frac{d[\tilde{x}]_S}{dt}\right]_S = [Q]_S$, $[U]_S = \left[\frac{dh}{dt}\right]_S$, $[a]_S = \left[\frac{dU}{dt}\right]_S = \left[\frac{d^2 h}{dt^2}\right]_S$.

Specific examples can now be worked out e.g. Let $[\tilde{x}]_S = [Q]_S$, with

$$[h]_S = [\tilde{x}(t)]_S = [r(t)]_S = [r_0 \{\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}\}]_S = [r_0 \hat{r}]_S$$

$$[\dot{h}]_S = [\dot{\tilde{x}}(t)]_S = [\dot{r}(t)]_S = -[r_0 \omega \{\sin(\omega t) \hat{x} - \cos(\omega t) \hat{y}\}]_S = [v]_S$$

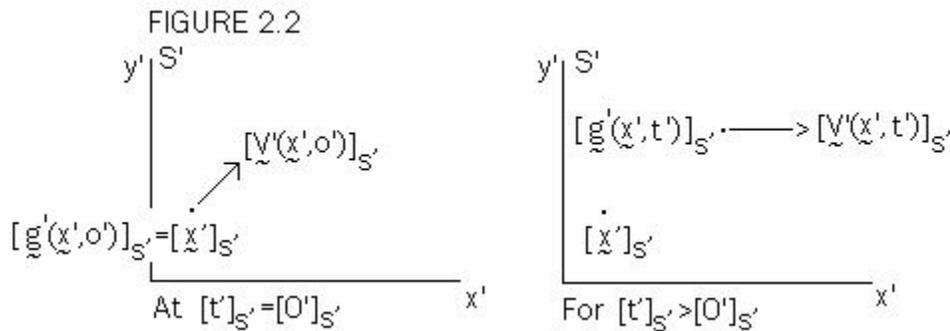
$$[\ddot{h}]_S = [\ddot{\tilde{x}}(t)]_S = [\ddot{r}(t)]_S = -[r_0 \omega^2 \{\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}\}]_S = -\left[\frac{v^2}{r_0} \hat{r}\right]_S = \text{the centripetal}$$

acceleration

2. Accelerating Point as Seen From S'

Consider a point moving arbitrarily within inertial frame S'. All determinations of position and time of the moving point are made by observers $O'_{x'}$ at rest in S'. Clocks in S' are synchronized with clocks in S in the usual way. Let $[x']_{S'}$ at rest in S' represent the position of a moving point at time $[t']_{S'}=0'$ and let the function $[g']_{S'} = ([g'_1]_{S'}, [g'_2]_{S'}, [g'_3]_{S'})$ represent the position of the moving point at arbitrary time $[t']_{S'}$. See figure 2.2.

$$2.3 \quad [g'(x',t')]_{S'} = [x']_{S'} + [\sigma'(x',t')]_{S'} \quad \text{with} \quad [\sigma'(x',0')]_{S'} = [Q']_{S'}$$



Repeating for the S' frame the derivation following 2.1 for the S frame yields

$$\left[\frac{dx'}{dt'}\right]_{S'} = [Q']_{S'}$$

$[g']_{S'} = [g'(x',t')]_{S'}$ with $\left[\frac{dx'}{dt'}\right]_{S'} = [Q']_{S'}$ will replace the symbol $[x'(t')]_{S'}$.

The velocity $[V']_{S'}$ of the arbitrarily moving point with respect to S' is: $[V']_{S'} = \left[\frac{dg'}{dt'}\right]_{S'}$

and the acceleration of the moving point is: $[a']_{S'} = \left[\frac{dV'}{dt'}\right]_{S'} = \left[\frac{d^2g'}{dt'^2}\right]_{S'}$ with:

$$2.4 \quad \left[\frac{\partial g'}{\partial t'}\right]_{S'} = \left[\frac{\partial \sigma'(x',t')}{\partial t'}\right]_{S'} \quad \text{and} \quad \left[\frac{\partial^2 g'}{\partial t'^2}\right]_{S'} = \left[\frac{\partial^2 \sigma'(x',t')}{\partial t'^2}\right]_{S'}$$

and in all cases, $\left[\frac{dx'}{dt'}\right]_{S'} = [Q']_{S'}$, $[V']_{S'} = \left[\frac{dg'}{dt'}\right]_{S'}$, $[a']_{S'} = \left[\frac{dV'}{dt'}\right]_{S'} = \left[\frac{d^2g'}{dt'^2}\right]_{S'}$.

3. Inertial Frame Transforms- Newton's Transform

The general linear transform 1.4, 1.5, must be re-examined to purge it of $\underline{x}=\underline{x}(t)$ and $\underline{x}'=\underline{x}'(t')$ where S and S' are inertial frames. The same criteria will be applied to any non inertial coordinate frame T.

In the following the physically correct linear transform, here named the Newtonian Transform, between two inertial frames S and S' is derived. S and S' are aligned with one another in the usual way and clocks in S are synchronized with clocks in S' by the method described in Chapter 1, section 2. The Galilean Transform 1.43, 1.44 will be examined first and then the general linear transform 1.4, 1.5.

Using the bracket notation, $[]_S$ means that the quantity in the bracket is observed by an observer fixed in S. For example $[t]_S=[t']_S$ means that the reading of clock $K_{\underline{x}}$ and the reading of clock $K'_{\underline{x}'}$ are equal as determined by $O_{\underline{x}}$ at rest at \underline{x} , at the moment of coincidence between \underline{x} and \underline{x}' . S' is moving with respect to S with velocity $[U]_S[\hat{x}]_S$. Consider two coincident points $[\underline{x}']_S$ and $[\underline{x}]_S$ at $[t]_S$ where $[\underline{x}']_S$ is stationary in S' and $[\underline{x}]_S$ is stationary in S. Rewriting the Galilean Transform for observers in S: Let $[k'_{x,t}]_S$ represent the $[x']_S$ component of point $[\underline{x}']_S$ fixed in S' whose position at time $[t]_S$ is coincident with point $[\underline{x}]_S$ fixed in S. In the Galilean Transform replace $[x']_S$ by $[k'_{x,t}]_S$.

$$2.5 \quad [k'_{x,t}]_S=[x]_S-[Ut]_S, [y']_S=[y]_S, [z']_S=[z]_S, \\ [t']_S=[t]_S$$

Writing 2.5 for observers stationary in S' using the bracket notation $[]_{S'}$ is left as a problem.

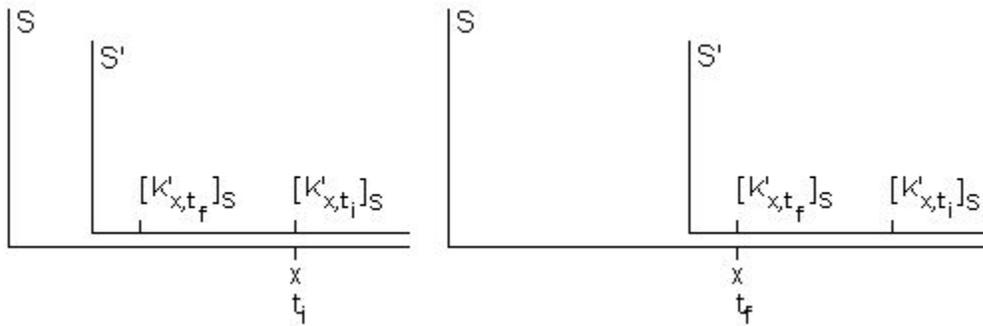
For a given fixed $[\underline{x}]_S$, $[k'_{x,t}]_S$ represents a unique point in S' for each instant of $[t]_S$. 2.5 represents the functional relationship between the coordinates of point $[k'_{x,t}]_S$ and the coordinates of $[\underline{x}]_S$ at the instant $[t]_S$ for which $[k'_{x,t}]_S$ is coincident with $[\underline{x}]_S$.

$[\underline{x}]_S$ is fixed in S so that $[\frac{d[\underline{x}]}{dt}]_S \equiv [0]_S$.

The speed of $[k'_{x,t}]_S$ with respect to S is $\left| \frac{d[k'_{x,t}]_S}{dt} \right| = \left| \lim_{t_f \rightarrow t_i} \frac{[k'_{x,t_i}]_S - [k'_{x,t_f}]_S}{t_f - t_i} \right| = |[U]_S| = [U]_S$

and the velocity of $[k'_{x,t}]_S$ with respect to S is $[U]_S[\hat{x}]_S$. See figure 2.3.

FIGURE 2.3



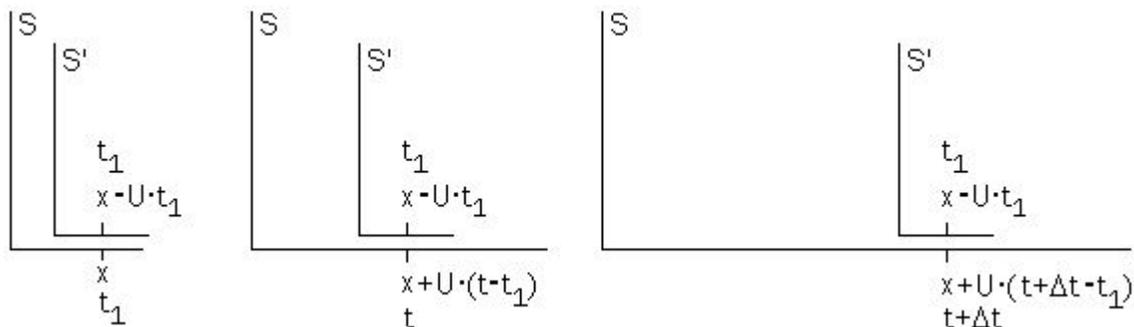
In order to determine the future position of a fixed point in S' with coordinates in S' $([k'_{x,t_1}]_S, [y']_S, [z']_S)$ at time t_1 as determined by an observer in S coincident with $([k'_{x,t_1}]_S, [y']_S, [z']_S)$ at time t_1 : Let $[j(x,t)_{k'_{x,t_1}}]_S$ represent the x -component in S of k'_{x,t_1} for $t \geq t_1 \geq 0$. $[x]_S$ is fixed in S so that $[\frac{d[x]_S}{dt}]_S = [0]_S$. The speed of $[j(x,t)_{k'_{x,t_1}}]_S$

$$2.6 \quad [j(x,t)_{k'_{x,t_1}}]_S = [x]_S + [U \cdot (t - t_1)]_S \quad t \geq t_1 \geq 0$$

with respect to S is $\frac{d[j(x,t)_{k'_{x,t_1}}]_S}{dt} = \lim_{\Delta t \rightarrow 0} \frac{[j(x,t+\Delta t)_{k'_{x,t_1}}]_S - [j(x,t)_{k'_{x,t_1}}]_S}{\Delta t} = [U]_S$ and the velocity of $[j(x,t)_{k'_{x,t_1}}]_S$ with respect to S is $[U]_S [\hat{x}]_S$. See figure 2.4.

Writing 2.6 for observers stationary in S' using the bracket notation $[]_{S'}$ is left as a problem.

FIGURE 2.4



2.5 and 2.6 taken together are the Newtonian Transform and are the replacement set of equations for the Galilean Transform. For reference:

2.7 Newtonian Transform

$$\begin{aligned}
 [k'_{x,t}]_S &= [x]_S - [Ut]_S, [y']_S = [y]_S, [z']_S = [z]_S, \\
 [j(x,t)_{k_{x,t_1}}]_S &= [x]_S + [U \cdot (t - t_1)]_S, [t]_S \geq [t_1] \geq [0] \\
 [t']_S &= [t]_S \\
 [U]_S &= \frac{d[j(x,t)_{k_{x,t_1}}]_S}{dt} = -\frac{d[k'_{x,t}]_S}{dt}
 \end{aligned}$$

A correctable error has been made by assuming the general linear transform between inertial frames S and S' is of the form 1.4 and 1.5. \underline{x} can represent either a point fixed in S or a point moving in S and \underline{x}' can represent either a point fixed in S' or a point moving in S'. To avoid ambiguity it is required, as in Chapter 2, Section 2 that \underline{x} represent a point fixed in S and \underline{x}' represent a point fixed in S'.

To be precise:

(i) x in 1.4 can represent the x component of stationary point \underline{x} fixed in S as determined by observers in S. This form is used in determining the x' position of stationary x at time t i.e. $x' = \alpha x + \beta t$. x is held stationary and t is allowed to vary. However x is also allowed to represent the x component of point \underline{x} moving with respect to S. i.e. $x = x(t)$ and $\frac{dx'}{dt'} = (\alpha \frac{dx}{dt} + \beta) \frac{dt}{dt'}$. $x(t)$ is however measured along the x axis of S where x represents a point stationary in S. The notation is inconsistent with itself.

(ii) x' and x as used in $x' = \alpha x + \beta t$ represent two different points at the moment of coincidence. x' is at rest in S' and x is at rest in S. Differentiating $x' = \alpha x + \beta t$ using $t' = \gamma x + \delta t$ yields $V' = \frac{(\alpha V + \beta)}{(\gamma V + \delta)}$ where $V' = \frac{dx'}{dt'}$ and $V = \frac{dx}{dt}$. The moving point to which V' refers and the moving point to which V refers are now one and the same point where the moving point has speed V' with respect to S' and speed V with respect to S. The notation is inconsistent with itself.

Correct this error by substituting $[k'_{x,t}]_S$ for x' in 1.4. This results in:

$$[k'_{x,t}]_S = [\alpha x]_S + [\beta t]_S \text{ and } [t']_S = [\gamma x]_S + [\delta t]_S \text{ and } [y']_S = [y]_S, [z']_S = [z]_S \text{ where } \left[\frac{dx}{dt}\right]_S \equiv 0.$$

Substituting $[k'_{x,t}]_S$ for x' in the proof of 1.42(b) yields: $\alpha = 1, \beta = -[U]_S, \gamma = 0, \delta = 1$.

The resulting transform is 2.5. Using $\alpha = 1, \beta = -[U]_S, \gamma = 0, \delta = 1$ the derivation of 2.6 follows with no change in nomenclature.

4. Accelerating Frame Transforms

Consider a non inertial frame T' accelerating with respect to an inertial frame S. Let the axis of S and T' remain parallel at all times. x parallel to x' , y parallel to y' , z parallel to z' . Clocks in S are synchronized with clocks in S in the usual way and clocks in T' are synchronized with clocks in S by setting a given running clock $K'_{\underline{x}}$ at

rest at \tilde{x}' in T' equal to the reading of running clock $K_{\tilde{x}}$ at rest at \tilde{x} in S at the instant of coincidence of $K_{\tilde{x}}$ with $K_{\tilde{x}'}$ as determined by observers $O_{\tilde{x}}$ and $O_{\tilde{x}'}$ respectively at rest at \tilde{x} and \tilde{x}' .

Using 2.1, it is assumed that the transform between S and T' as determined by observers in S is given by the immediate generalization of the Newtonian Transform

$$2.7. \text{ From 2.5: } [k'_{\tilde{x},t}]_S = [\tilde{x}]_S - \left[\int_0^t [U]_S dt \right]_S, \quad [t']_S = [t]_S, \quad \text{and } [U(t)]_S = \left[-\frac{d}{dt} [k'_{\tilde{x},t}]_S \right]_S = \left[-\frac{d}{dt} k'_{\tilde{x},t} \right]_S$$

and $[a(t)]_S = \left[\frac{d}{dt} \left[-\frac{d}{dt} [k'_{\tilde{x},t}]_S \right]_S \right]_S = \left[-\frac{d^2}{dt^2} k'_{\tilde{x},t} \right]_S$. All points in S' move with the same

velocity $[U(t)]_S$ and acceleration $[a(t)]_S$ with respect to S . $[k'_{\tilde{x},t}]_S = [\tilde{x}]_S - \left[\int_0^t [U]_S dt \right]_S$ is used to determine the S' coordinates of the point $[k'_{\tilde{x},t}]$ fixed in S' that is coincident with point \tilde{x} fixed in S at time t as determined by $O_{\tilde{x}}$. For reference:

$$2.8 \quad \begin{aligned} [k'_{\tilde{x},t}]_S &= [\tilde{x}]_S - \left[\int_0^t [U]_S dt \right]_S \\ [t']_S &= [t]_S \\ [U(t)]_S &= \left[-\frac{d}{dt} k'_{\tilde{x},t} \right]_S, \quad [a(t)]_S = \left[-\frac{d^2}{dt^2} k'_{\tilde{x},t} \right]_S \end{aligned}$$

Writing 2.8 for observers stationary in T' using the bracket notation $[]_{T'}$ is left as a problem.

In order to determine the future position of a fixed point $[k'_{\tilde{x},t_1}]_S$ in T' with coordinates $[k'_{\tilde{x},t_1}]_S$ in T' at time t_1 as determined by an observer in S coincident with $[k'_{\tilde{x},t_1}]_S$ at time t_1 : Let $[j(\tilde{x},t)_{k'_{\tilde{x},t_1}}]_S$ represent the position in S of $k'_{\tilde{x},t_1}$ for $t \geq t_1 \geq 0$:

$$[j(\tilde{x},t)_{k'_{\tilde{x},t_1}}]_S = [\tilde{x}]_S + \left[\int_{t_1}^t [U]_S dt \right]_S. \quad [\tilde{x}]_S \text{ is fixed in } S \text{ so that } \left[\frac{d[\tilde{x}]_S}{dt} \right]_S = [0]_S. \text{ The velocity}$$

$$[U(t)]_S \text{ of } [j(\tilde{x},t)_{k'_{\tilde{x},t_1}}]_S \text{ with respect to } S \text{ is } [U(t)]_S = \left[\frac{d}{dt} [j(\tilde{x},t)_{k'_{\tilde{x},t_1}}]_S \right]_S = \left[\frac{d}{dt} j(\tilde{x},t)_{k'_{\tilde{x},t_1}} \right]_S$$

$$\text{and the acceleration } [a(t)]_S \text{ is } [a(t)]_S = \left[\frac{d}{dt} [U(t)]_S \right]_S = \left[\frac{d}{dt} \left[\frac{d}{dt} [j(\tilde{x},t)_{k'_{\tilde{x},t_1}}]_S \right]_S \right]_S = \left[\frac{d^2}{dt^2} j(\tilde{x},t)_{k'_{\tilde{x},t_1}} \right]_S.$$

All points in S' move with the same velocity $[U(t)]_S$ and acceleration $[a(t)]_S$ with respect to S . For reference:

$$2.9 \quad [j(\underline{x}, t)_{k'_{\underline{x}, t_1}}]_S = [\underline{x}]_S + [\int_{t_1}^t [\underline{U}]_S dt]_S \quad t \geq t_1 \geq 0$$

$$[\underline{U}(t)]_S = [\frac{d}{dt} j(\underline{x}, t)_{k'_{\underline{x}, t_1}}]_S, \quad [a(t)]_S = [\frac{d^2}{dt^2} j(\underline{x}, t)_{k'_{\underline{x}, t_1}}]_S$$

Writing 2.6 for observers stationary in S' using the bracket notation $[]_{S'}$ is left as a problem.

2.8 and 2.9 taken together are the Generalized Newtonian Transform. For reference:

2.10 Generalized Newtonian Transform

$$[k'_{\underline{x}, t}]_S = [\underline{x}]_S - [\int_0^t [\underline{U}]_S dt]_S$$

$$[j(\underline{x}, t)_{k'_{\underline{x}, t_1}}]_S = [\underline{x}]_S + [\int_{t_1}^t [\underline{U}]_S dt]_S \quad t \geq t_1 \geq 0$$

$$[t']_S = [t]_S$$

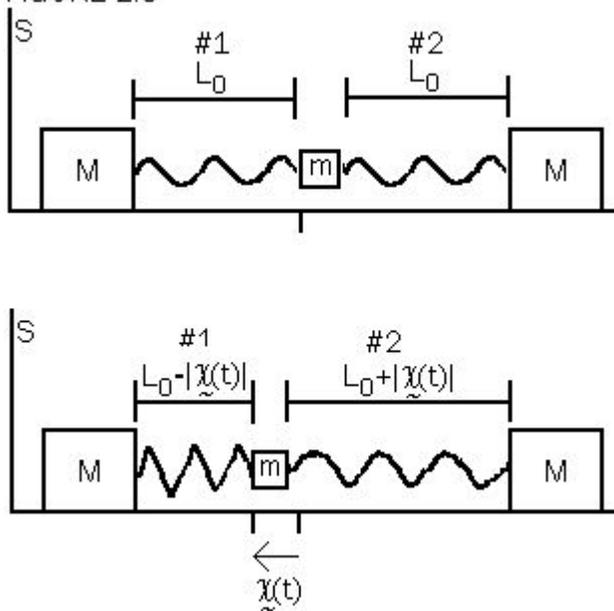
$$[\underline{U}(t)]_S = [-\frac{d}{dt} k'_{\underline{x}, t}]_S = [\frac{d}{dt} j(\underline{x}, t)_{k'_{\underline{x}, t_1}}]_S, \quad [a(t)]_S = [-\frac{d^2}{dt^2} k'_{\underline{x}, t}]_S = [\frac{d^2}{dt^2} j(\underline{x}, t)_{k'_{\underline{x}, t_1}}]_S$$

5. Operational Definition of Inertial Mass

An operational definition of inertial mass is given independent of the operational definition of gravitational mass.

Consider two identical springs #1 and #2 each of which is attached to a mass M . See figure 2.5. A third mass, mass m , is attached to the free end of springs #1 and #2 and the ensemble is placed at rest in inertial frame S . No subscripts are used as all measurements are made by observers at rest with respect to inertial frame S . The rest length of both springs when far from all masses is L_0 . The configuration of figure 2.5 is chosen so that the gravitational force of the M masses on m cancel one another.

FIGURE 2.5



Due to the gravitational attraction of the M masses for one another, each spring will be compressed an amount ΔL_0 . M , m and the spring stiffness are chosen so that $\Delta L_0 \ll L_0$. It is assumed that the dimensions of M and m remain constant during the acceleration process. $\tilde{x}(t)$ is defined as $\tilde{x}(t) \equiv [L(t) - L_0] \hat{x}$ where $L(t)$ is the length of spring #1 at time t and $L(0) \equiv L_1$ and $\tilde{x}(0) = [L_1 - L_0] \hat{x}$.

Spring #1 is compressed to length L_1 and spring #2 is expanded to length $2L_0 - L_1$ and both springs are held to that length. The mass M is large enough so that when the springs are released, the displacement of the M masses is unobservable and all observable displacements are of mass m . The springs are chosen so as to obey Hook's Law: $m\ddot{\tilde{x}} = -2K\tilde{x}(t)$ or $\ddot{\tilde{x}} = -\frac{2K}{m}\tilde{x}(t)$, $\tilde{x}(0) = [L_1 - L_0] \hat{x}$, $\dot{\tilde{x}}(0) = 0$, with solution

$$\tilde{x}(t) = \cos \omega t \tilde{x}(0), \quad \omega^2 = \frac{2K}{m}.$$

Unit mass m_0 is chosen to be 1cm^3 of water at 4°C with temperature measured using a mercury thermometer. The unit mass is called the gram and $m_0 \equiv 1\text{gm}$. The water is frozen and attached to the free end of springs #1 and #2 and the frequency ω_0 is directly measured. The spring constant Kgm/sec^2 is now determined using $K = \frac{\omega_0^2 m_0}{2}$. An unknown mass m_1 may now be determined by attaching m_1 to the

spring ensemble and directly measuring ω_1 . m_1 is determined from $m_1 = \frac{2K}{\omega_1^2} = \frac{\omega_0^2}{\omega_1^2} m_0$.

Operationally defined mass m can now be used with operationally defined acceleration to operationally define force as $[\tilde{F}]_S = [m]_S [\tilde{\ddot{x}}]_S$ e.g. Using the expression for centripetal acceleration following 2.2, centripetal force is defined as

$$[\tilde{F}]_S = -[m]_S \left[\frac{v^2}{r_0} \hat{r} \right]_S.$$

A force scale is now constructed from one spring and calibrated using $\tilde{F} = -K[L - L_0] \hat{x}$ where L is the measured length of the spring under either tension or compression.

6. Operational Definition of Gravitational Mass

Using the scale constructed in Section 5, recalibrate the scale as follows so as to measure gravitational mass. Suspend the spring from one end on a suitable frame and place the frame at rest at the North or South Pole on the earth's surface (Due account being made of the fact that the North and South Pole are not the same distance from the center of the earth). A coordinate frame is chosen at rest with respect to the earth's surface. The frame is not an inertial frame and due account must be taken of that fact when measuring gravitational mass and acceleration. The spring is now suspended in the vertical and the unit inertial mass m_0 defined in section 5 is attached to the free end of the spring. With L_0 the rest length of the spring in an inertial frame (The same spring of section 5), let L_{0g} be the rest length of the spring suspended in the vertical in the earth's gravitational field. See figure 2.6. $\chi_{m_0,g} \equiv [L_{m_0,g} - L_{0g}]$ is measured with m_0 suspended on the spring and at rest with respect to the earth. Any object attached to the scale with resultant $\chi_{m_0,g}$ has a gravitational mass of unity. It has not been shown that unit inertial mass m_0 is equal to unit gravitational mass $m_{0,g}$. n unit masses are attached and $\chi_{nm_0} \equiv [L_{nm_0} - L_{0g}]$ is measured... In this way a recalibrated scale can be constructed enabling one to measure an unknown gravitational mass m_g in terms of the length of the stretched spring. It is assumed that the spring in a gravitational field is still a Hook's Law spring although perhaps with a different spring constant K_1 and rest length L_{0g} .

$\tilde{F}_g = m_g g \hat{z} = K_1 [L_{m,g} - L_{0g}] \hat{z} \equiv K_1 \chi_{m,g} \hat{z}$ where \tilde{F}_g is the force on m_g due to the spring and g is the measured local acceleration of gravity, $g = 981 \text{ cm/sec}^2$. With m_{0g} = unit gravitational mass, $m_{0g} g \hat{z} = K_1 \chi_{m_{0g}} \hat{z}$. K_1 is determined with unit gravitational mass in place where $K_1 = m_{0g} g / \chi_{m_{0g}}$. The unknown gravitational mass m_g of an object A is given by $m_g = m_{0g} \chi_{m,g} / \chi_{m_{0g}}$.

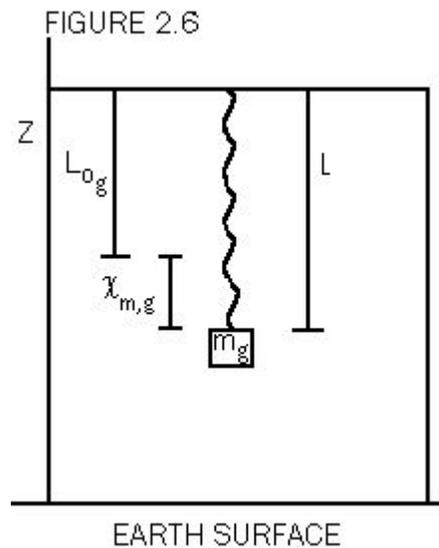
If $K_1=K$, i.e. if $g/\chi_{m_0,g} = \frac{\omega_0^2}{2}$, then $m_{0,g}g = K_1\chi_{m_0,g} = K\chi_{m_0,g}$. With m_0 in the inertial

spring ensemble fig. 2.5, let $\chi(0) = -\frac{\chi_{m_0,g}}{2} \hat{x}$ and $\dot{\chi}(0) = 0$. From chapter 6, section 5:

$m_0 \ddot{\chi}(0) = -2K\chi(0) = K\chi_{m_0,g}$ and using $m_{0,g}g = K\chi_{m_0,g}$ yields $m_{0,g}g = m_0 \ddot{\chi}(0)$. If

experimentally $g = \ddot{\chi}(0)$, then $m_{0,g} = m_0$ i.e. Gravitational Mass = Inertial Mass.

The Eötvös experiment supports the assumption that Gravitational Mass = Inertial Mass and in the rest of this text, it will be assumed that Gravitational Mass = Inertial Mass. See reference 2.4 .



7. The Lagrange and Hamilton Equations

It is suggested that the reader review the derivation of the Lagrangian and the Hamiltonian as given in reference 2.5, before reading this section.

Consider a system of p point like particles with $3p$ position coordinates interacting with one another via field forces. The position of the i th particle with coordinates x_i, y_i, z_i is dependent on the position of the remaining $p-1$ particles and is usually written as:

$$x_i = x_i(x_1, y_1, z_1, \dots, x_j, y_j, z_j, \dots, x_p, y_p, z_p), \quad y_i = y_i(x_1, y_1, z_1, \dots, x_j, y_j, z_j, \dots, x_p, y_p, z_p),$$

$$z_i = z_i(x_1, y_1, z_1, \dots, x_j, y_j, z_j, \dots, x_p, y_p, z_p), \quad j \neq i \text{ where } x_j = x(t)_j, \quad y_j = y(t)_j, \quad z_j = z(t)_j \text{ with } j=1, 2, \dots, p.$$

However as discussed above, the notation $x_j = x(t)_j, y_j = y(t)_j, z_j = z(t)_j$ is inconsistent with itself. To avoid this inconsistency replace x_j with q_{3j-2} and y_j with q_{3j-1} and z_j with q_{3j} where $q_{3j-2} = q(x_j, t)_{3j-2}, q(x_j, 0)_{3j-2} = x_j$ and $q_{3j-1} = q(y_j, t)_{3j-1}, q(y_j, 0)_{3j-1} = y_j$

and $q_{3j} = q(z_j, t)_{3j}, q(z_j, 0)_{3j} = z_j$ where x_j, y_j, z_j are fixed points in inertial frame S . The q_k with $k=1, 2, \dots, 3p$ represent the $3p$ position coordinates of the p particles.

It is assumed that a potential function V exists with $V = V(q_1, q_2, \dots, q_{3p})$ such that the x, y, z

component of the force acting on the j th particle is $F_x^{(q_{3j-2})} = -\frac{\partial V}{\partial q_{3j-2}}$ with $F_y^{(q_{3j-1})} = -\frac{\partial V}{\partial q_{3j-1}}$

and $F_z^{(q_{3j})} = -\frac{\partial V}{\partial q_{3j}}$. The kinetic energy of the p particles is defined as the sum

$$T = \frac{1}{2} \sum_{j=1}^p m_j (\dot{q}_{3j-2}^2 + \dot{q}_{3j-1}^2 + \dot{q}_{3j}^2) \text{ and the Lagrangian function is defined as } L = T - V.$$

Assuming L exists as stated above: Hamilton's Principle is that the actual motion of a system is such as to render the integral an extremum with respect to continuously twice differentiable functions q_k , $k=1,2,\dots,3p$ where p is the number of particles

$$2.11 \quad I = \int_{t_1}^{t_2} L dt$$

and where t_1 and t_2 are two arbitrary instants of time.

The Lagrangian can be shown to satisfy the Lagrange Equation (Ref. 2.5):

$$2.12 \quad \frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0 \text{ with } k=1,2,\dots,3p. \quad L = L(q_1, q_2, \dots, q_k, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k)$$

$$a. \quad q_k = q\left(\frac{x_{k+2}}{3}, t\right)_k, \quad q_k = q\left(\frac{x_{k+2}}{3}, 0\right)_k = \frac{x_{k+2}}{3}, \quad k=1,4,7,\dots,3p-2$$

$$b. \quad q_k = q\left(\frac{y_{k+1}}{3}, t\right)_k, \quad q_k = q\left(\frac{y_{k+1}}{3}, 0\right)_k = \frac{y_{k+1}}{3}, \quad k=2,5,8,\dots,3p-1$$

$$c. \quad q_k = q\left(\frac{z_k}{3}, t\right)_k, \quad q_k = q\left(\frac{z_k}{3}, 0\right)_k = \frac{z_k}{3}, \quad k=3,6,9,\dots,3p$$

It can be shown that the total energy E of the p particles is a constant of the motion where $E = T + V$.

It has been shown that the q_k which are called generalized coordinates are the Cartesian Coordinates expressed in a self consistent form. i.e. $x_j \neq x(t)_j$, $y_j \neq y(t)_j$, $z_j \neq z(t)_j$ and x_j, y_j, z_j replaced by q_k as given by 2.12- a,b,c.

With in the context of the Lagrangian given above, define the set p_1, p_2, \dots, p_{3p} of generalized momenta.

$$2.13 \quad p_k = \frac{\partial T}{\partial \dot{q}_k}, \quad k=1,2,\dots,3p \text{ where } p \text{ is the number of particles}$$

Using 2.13 eliminate the \dot{q}_k from the Lagrangian so that $L = L(q_1, q_2, \dots, q_{3p}, p_1, p_2, \dots, p_{3p})$.

The Hamiltonian is defined as:

$$2.14 \quad H(q_1, q_2, \dots, q_{3p}, p_1, p_2, \dots, p_{3p}) = \sum_{j=1}^{3p} p_j \dot{q}_j - L = \sum_{j=1}^{3p} \frac{p_j^2}{m_j} - L(q_1, q_2, \dots, q_{3p}, \frac{p_1}{m_1}, \frac{p_2}{m_2}, \dots, \frac{p_{3p}}{m_{3p}})$$

where $m_j = m_{j+1} = m_{j+2}$ for $j=1, 4, 7, \dots, 3p-2$

2.13 has been used to eliminate the \dot{q}_k from the right hand side of 2.14. Taking derivatives of H with respect to p_k and q_k yields: $\frac{\partial H}{\partial p_k} = 2 \frac{p_k}{m_k} = \frac{\partial T}{\partial p_k} = 2 \frac{p_k}{m_k} = \frac{p_k}{m_k} = \dot{q}_k$ and $\frac{\partial H}{\partial q_k} = \frac{\partial V}{\partial q_k} = -\dot{p}_k = -m_k \ddot{q}_k$

$$2.15 \quad \frac{\partial H}{\partial p_k} = \dot{q}_k, \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k, \quad K=1, 2, \dots, 3p$$

2.15 are Hamilton's Equations. Using 2.14 and the definition of T:

$$H = \sum_{j=1}^{3p} \frac{p_j^2}{m_j} - L = 2T - (T - V) = T + V = \text{const. of the motion.}$$

It has been shown that the q_k which are called generalized coordinates are the Cartesian Coordinates expressed in a self consistent form. i.e. $x_j \neq x(t)_j$, $y_j \neq y(t)_j$, $z_j \neq z(t)_j$ and x_j, y_j, z_j replaced by q_k as given by 2.12- a,b,c.

8. Newton's Second Law

A finite mass sample of a solid, liquid or gas is made up of a finite number of finite volume, continuous mass particles, (Atoms and photons), each of which is separated by empty space. Newton's Laws were originally written for mass points. Mass points do not exist and consequently Newton's Laws must be rewritten for finite volume, continuous mass particles. This is done in chapter 6, section 4.

The Navier-Stokes Equations are an attempt to write Newton's Second Law for continuous medium. However in the derivation of the Navier- Stokes Equation, Reference 2.1, the erroneous assumption is made that $[\underline{x}]_S = [\underline{x}(t)]_S$. By straightforward differentiation erroneously assuming $[\underline{U}]_S = [\underline{U}(\underline{x}(t), t)]_S$, it follows that

$[\underline{a}]_S = \left[\frac{d\underline{U}}{dt} \right]_S = \left[\frac{\partial \underline{U}}{\partial t} \right]_S + [(\underline{U} \cdot \nabla) \underline{U}]_S$. The $[(\underline{U} \cdot \nabla) \underline{U}]_S$ term is used in the derivation of the oceanic tidal amplitude and in a description of fluid turbulence. However $[\underline{x}]_S = [\underline{x}(t)]_S$

does not represent physical reality and $[\underline{a}]_S = \left[\frac{d\underline{U}}{dt} \right]_S = \left[\frac{\partial \underline{U}}{\partial t} \right]_S$ represents physical

reality while $[\underline{a}]_S = \left[\frac{d\underline{U}}{dt} \right]_S = \left[\frac{\partial \underline{U}}{\partial t} \right]_S + [(\underline{U} \cdot \nabla) \underline{U}]_S$ does not represent physical reality: See

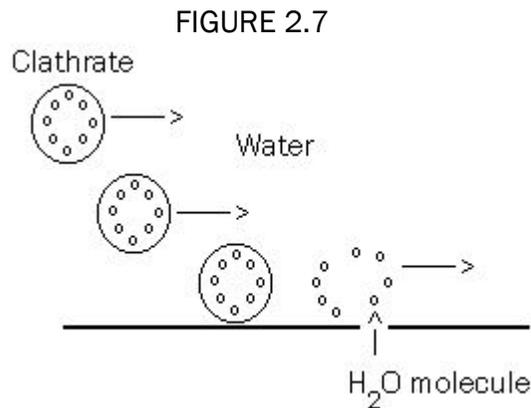
Chap. 2, Sec. 1 . A new explanation of the oceanic tides and fluid turbulence is therefore necessary.

The oceanic tides can be shown to be due to the gravitational attraction between the moon, the sun and the liquid interior of the earth: See chap 11, Sec. 8 and 9. The liquid interior under the influence of the moon and sun push up on the underside of the oceanic lithosphere causing the oceanic lithosphere to move in the vertical some 0 to 40m depending on latitude. The ocean flows down the flanks of the formed bulges and the water flowing down the flanks of the bulges cause the oceanic tide. Ref. 2.2

As regards turbulent flow.

The transition from laminar flow to turbulent flow is hypothesized to result when shear forces are created in the fluid which are large enough to break up fluid clathrates into the individual molecules of which the clathrate is composed. Turbulent flow results. In many cases, the break up of the clathrates occurs on a gas-solid or liquid-solid boundary. Reference 3.

For example, water molecules are held together by inter molecular bonds forming a supra molecular clathrate structure. It is hypothesized that if the water clathrate strikes a solid stationary retaining structure with sufficient velocity, the clathrate will break apart into its component water molecules and that is the physical cause of turbulent flow. Figure 2.7 .



If the clathrate is composed of n H₂O molecules whose average kinetic energy K.E. is: $K.E. = \frac{1}{2} n m_{H_2O} v^2 = \frac{3}{2} KT$, then after the breakup and upon reaching equilibrium temperature, each water molecule will have $K.E. = \frac{1}{2} m_{H_2O} U^2 = \frac{3}{2} KT$, and the local pressure will increase from a background pressure of P_0 to a local pressure of nP_0 . A new theory for shock waves and fluid turbulence is developed in chapter 12.

References

- 2.1 Sir H. Lamb, Hydrodynamics, Chapter 1, (Dover Publications, New York, 1945)
- 2.2 B. Fisher, Letter, (May 16, 1997)
- 2.3 T. Gurney, Letter, (March 16, 1973)
- 2.4 E.H. Wichmann, Notes for Physics 231, Chapter 9, Theory of Relativity and Cosmology, Spring 1973, University of California, Berkeley
- 2.5 R. Weinstock, Calculus of Variations, Chapter 6, (McGraw-Hill Book Company, New York, 1952)